

Heisenberg algebra for restricted Landau problem

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Abstract

Algebraic derivation of modified Heisenberg commutation rules for restricted Landau problem is given.

*supported by the Łódź University grant No. 690.

It appears that the noncommutative geometry should play an important role in our attempts to understand the structure of space-time at short distances/high energies. Therefore, it seems reasonable to study how the fundamental notions of noncommutative geometry arise in simple physical settings. In particular, much attention has been paid to problem of motion of charged particles in two dimensions under the influence of constant perpendicularly applied magnetic field, [1] - [7] (see also [8]). The non-commutativity of coordinates appears here naturally after restricting the system to the lowest Landau level; for the most recent discussion see [9]. This can be seen either by skipping the kinetic term in the lagrangian and quantizing the resulting constrained system by Dirac method or by saturating the commutation rule for coordinates by the states belonging to the lowest Landau level only. The generalization to the higher Landau level has been also given [10], [11]. It appears that, when higher levels are included the commutator of coordinates is no longer a c-number.

In the present brief note we give a general, purely algebraic derivation of commutation rules for basic canonical variables for the theory restricted to arbitrary finite number of Landau levels. The only things we are using are the Heisenberg commutation relations and the form of the energy spectrum.

Our starting point is the hamiltonian describing the planar motion of mass m charged particle in perpendicular magnetic field B

$$H = \frac{1}{2m}(p_i + \frac{eB}{2c}\varepsilon_{ik}x_k)^2; \quad (1)$$

here we have chosen the rotational invariant gauge, $A_i = -\frac{B}{2}\varepsilon_{ik}x_k$. The spectral decomposition of H reads

$$H = \sum_{n=0}^{\infty} \hbar\omega(n + \frac{1}{2})P_n \quad (2)$$

where $\omega = \frac{eB}{mc}$ and P_n is the projector on n -th level (which is infinite-dimensional subspace). P_n can be expressed in terms of H as follows:

$$P_n = \frac{4}{\pi(2n+1)} \frac{\sin\left((n + \frac{1}{2})\pi(\frac{H}{(n+\frac{1}{2})\hbar\omega} - 1)\right)}{(\frac{H}{(n+\frac{1}{2})\hbar\omega} - 1)(\frac{H}{(n+\frac{1}{2})\hbar\omega} + 1)} \quad (3)$$

One can easily check that the r. h. s. of eq. (3) is well defined. In what follows we shall use some simple relations listed below:

$$\begin{aligned} [H, x_i] &= -\frac{i\hbar}{m}(p_i + \frac{m\omega}{2}\varepsilon_{ik}x_k) \equiv -i\hbar\dot{x}_i \\ [H, [H, x_j]] &= -i\hbar\omega\varepsilon_{jk}[H, x_k] \\ [H, p_i - \frac{m\omega}{2}\varepsilon_{ik}x_k] &= 0 \\ [[H, x_k], [H, x_l]] &= -\frac{i\hbar^3}{m}\omega\varepsilon_{kl} \\ \sum_{k=1}^2 [H, x_k][H, x_k] &= -\frac{2\hbar^2}{m}H \end{aligned} \quad (4)$$

Note, that the motion described by the hamiltonian (1) is purely harmonic with frequency ω . Therefore, if A is any operator linear in x_i 's and p_i 's, $[H, A]$ connects only the eigenstates of H differing by $\hbar\omega$, i.e. the only nonvanishing operators are $P_n[H, A]P_{n+1}$ and $P_{n+1}[H, A]P_n$. The followig relation is a direct consequence of the last eq. (4)

$$\sum_{k=1}^2 P_n[H, x_k](P_{n+1} + P_{n-1})[H, x_k]P_n = -\frac{2\hbar^3\omega}{m}(n + \frac{1}{2})P_n \quad (5)$$

On the other hand note that

$$\begin{aligned} & \sum_{k=1}^2 P_n[H, x_k](\hbar\omega(n + \frac{3}{2})P_{n+1} + \hbar\omega(n - \frac{1}{2})P_{n-1})[H, x_k]P_n = \\ &= \sum_{k=1}^2 P_n[H, x_k]H[H, x_k]P_n = \\ &= \sum_{k=1}^2 P_n[H, x_k][H, [H, x_k]]P_n - \frac{2\hbar^4\omega^2}{m}(n + \frac{1}{2})^2P_n = \\ &= -i\hbar\omega \sum_{k,l=1}^2 \varepsilon_{kl}P_n[H, x_k][H, x_l]P_n - \frac{2\hbar^4\omega^2}{m}(n + \frac{1}{2})^2P_n = \\ &= -\frac{i\hbar\omega}{2} \sum_{k,l=1}^2 \varepsilon_{kl}P_n[[H, x_k], [H, x_l]]P_n - \frac{2\hbar^4\omega^2}{m}(n + \frac{1}{2})^2P_n = \\ &= -\frac{2\hbar^4\omega^2}{m} \left(\frac{1}{2} + (n + \frac{1}{2})^2 \right) P_n \end{aligned} \quad (6)$$

From eqs. (5) and (6) one finds

$$\begin{aligned} & \sum_{k=1}^2 P_n[H, x_k]P_{n+1}[H, x_k]P_n = -\frac{\hbar^3\omega}{m}(n + 1)P_n \\ & \sum_{k=1}^2 P_n[H, x_k]P_{n-1}[H, x_k]P_n = -\frac{\hbar^3\omega}{m}nP_n \end{aligned} \quad (7)$$

Finally, we shall need the following relations

$$\begin{aligned} [f(H), B]P_n &= \frac{f(H) - f((n + \frac{1}{2})\hbar\omega)}{H - (n + \frac{1}{2})\hbar\omega}[H, B]P_n \\ P_n[f(H), B] &= P_n[H, B] \frac{f(H) - f((n + \frac{1}{2})\hbar\omega)}{H - (n + \frac{1}{2})\hbar\omega} \end{aligned} \quad (8)$$

which hold for any operator B and any analytic function f . They can be proven by expanding $f(H)$ in power series and using $P_n H = H P_n = \hbar\omega(n + \frac{1}{2})P_n$.

Let us now consider the theory obtained from the hamiltonian (1) by imposing the cutoff $E \leq \hbar\omega(N + \frac{1}{2})$. Let $\Pi_N = \sum_{n=0}^N P_n$ be the projection operator on the relevant subspace. For any operator B let

$$\hat{B} = \Pi_N B \Pi_N \quad (9)$$

be its counterpart in cut-off theory. We are interested in commutation rules between cut-off operators; in particular, we would like to calculate the commutation rules between basic dynamical variables. Let us start with $[\hat{x}_i, \hat{x}_j]$. Using $[x_i, x_j] = 0$ and $\Pi_N[x_i, \Pi_N]\Pi_N = 0$ we find

$$[\hat{x}_i, \hat{x}_j] = \Pi_N[\Pi_N, x_i][\Pi_N, x_j]\Pi_N - (i \leftrightarrow j) \quad (10)$$

From eq. (8) one gets

$$[\Pi_N, x_i]\Pi_N = \frac{1}{H - (N + \frac{1}{2})\hbar\omega}(\Pi_N - 1)[H, x_i]P_N \equiv g(H)[H, x_i]P_N \quad (11)$$

so that eq. (10) can be rewritten as

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= P_N[H, x_i]g^2(H)[H, x_j]P_N - (i \leftrightarrow j) = \\ &= P_N [[H, x_i], [H, x_j]] g^2(H)P_N + (P_N[H, x_i][g^2(H), [H, x_j]]P_N - (i \leftrightarrow j)) \end{aligned} \quad (12)$$

In the first term on r. h. s. of eq. (12) we find

$$g^2(H)P_N = g^2\left((N + \frac{1}{2})\hbar\omega\right)P_N = \frac{1}{(2N + 1)^2\hbar^2\omega^2}P_N \quad (13)$$

while for the second term we use again eq. (8) to get

$$\begin{aligned} P_N[H, x_i] \frac{g^2(H) - g^2((N + \frac{1}{2})\hbar\omega)}{H - (N + \frac{1}{2})\hbar\omega} [H, [H, x_j]]P_N - (i \leftrightarrow j) &= \\ = i\hbar\omega\varepsilon_{ij} \sum_{k=1}^2 P_N[H, x_k] \frac{g^2(H) - g^2((N + \frac{1}{2})\hbar\omega)}{H - (N + \frac{1}{2})\hbar\omega} [H, x_k]P_N \end{aligned} \quad (14)$$

Inserting $I = \sum_{n=0}^{\infty} P_n$ one can write the last expression in the form

$$\begin{aligned} i\hbar\omega\varepsilon_{ij} \sum_{k=1}^2 P_N[H, x_k] \left(\frac{g^2((N + \frac{3}{2})\hbar\omega) - g^2((N + \frac{1}{2})\hbar\omega)}{\hbar\omega} P_{N+1} + \right. \\ \left. + \frac{g^2((N - \frac{1}{2})\hbar\omega) - g^2((N + \frac{1}{2})\hbar\omega)}{-\hbar\omega} P_{N-1} \right) [H, x_k]P_N = \\ = -\frac{i\hbar\varepsilon_{ij}}{m\omega(2N + 1)^2} (4N(N + 1)^2 + N)P_N \end{aligned} \quad (15)$$

where we have used (7) and (11).

Collecting all equations starting from eq. (12) we arrive at the following result

$$[\hat{x}_i, \hat{x}_j] = -\frac{i\hbar}{m\omega}(N + 1)\varepsilon_{ij}P_N \quad (16)$$

In a similar way one can compute the remaining basic commutation rules

$$\begin{aligned} [\hat{p}_i, \hat{p}_j] &= -\frac{i\hbar m\omega}{4}(N + 1)\varepsilon_{ij}P_N \\ [\hat{x}_i, \hat{p}_j] &= i\hbar(1 - \frac{1}{2}(N + 1))\delta_{ij}P_N \end{aligned} \quad (17)$$

Eqs. (16), (17) provide the basic algebra for cut-off theory. Keeping in mind that P_N is given by eq. (3) we see that the Heisenberg algebra is in this case a kind of W -algebra. The validity of the algebra (16)-(17) can be checked by explicit construction of the space of states [12]. We define the annihilation and creation operators a_{\pm} , a_{\pm}^{\dagger} :

$$\begin{aligned} a_+ &= \frac{1}{\sqrt{2m\omega\hbar}}(p_1 + ip_2) - \frac{i}{2}\sqrt{\frac{m\omega}{2\hbar}}(x_1 + ix_2) \\ a_- &= \frac{1}{\sqrt{2m\omega\hbar}}(p_1 - ip_2) - \frac{i}{2}\sqrt{\frac{m\omega}{2\hbar}}(x_1 - ix_2) \\ a_+^{\dagger} &= \frac{1}{\sqrt{2m\omega\hbar}}(p_1 - ip_2) + \frac{i}{2}\sqrt{\frac{m\omega}{2\hbar}}(x_1 - ix_2) \\ a_-^{\dagger} &= \frac{1}{\sqrt{2m\omega\hbar}}(p_1 + ip_2) - \frac{i}{2}\sqrt{\frac{m\omega}{2\hbar}}(x_1 + ix_2) \end{aligned} \quad (18)$$

The hamiltonian (1) takes now the form

$$H = \hbar\omega(a_+^{\dagger}a_+ + \frac{1}{2}) \quad (19)$$

while the angular momentum reads

$$L = \hbar(a_-^{\dagger}a_- - a_+^{\dagger}a_+) \quad (20)$$

We see that a_- , a_-^{\dagger} produce the states of the same energy but different angular momentum.

Our cut-off space of states, (see, also, [13]), is spanned by the vectors $|n_+, n_->$, $n_+ = 0, 1, \dots, N$, $n_- = 0, 1, 2, \dots$. The modified commutation rules read

$$\begin{aligned} [a_-, a_-^{\dagger}] &= 1 \\ [a_+, a_+^{\dagger}] &= 1 - (N+1)P_N \\ [a_+, a_-] &= [a_+^{\dagger}, a_-^{\dagger}] = [a_+^{\dagger}, a_-] = [a_+, a_+^{\dagger}] = 0 \end{aligned} \quad (21)$$

and

$$P_N |n_+, n_-> = \delta_{N, n_+} |n_+, n_-> \quad (22)$$

Using eqs. (18) and (21) one easily verifies the validity of commutation rules (16) and (17).

Acknowledgment

We would like to thank prof. P. Kosiński for interesting discussions.

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